1.HIGHLIGHTS OF ELLIPSE :

Referring to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

H-1 If P be any point on the ellipse with S & S′ as its foci then $\ell(SP) + \ell(S'P) = 2a$.

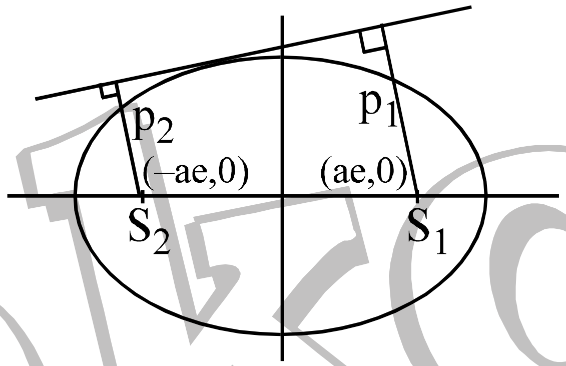
H-2

(a) Product of the length's of the perpendiculars from either focus on a variable tangent to an Ellipse / Hyperbola = (semi minor axis)^2 / (semi conjugate axis)^2 = b^2

$mx - y + \sqrt{a^2m^2 + b^2} = 0$

$$p_1p_2 = \left| \frac{mae + \sqrt{a^2m^2+b^2}}{\sqrt{1+m^2}} \cdot \frac{-mae + \sqrt{a^2m^2+b^2}}{\sqrt{1+m^2}} \right|$$

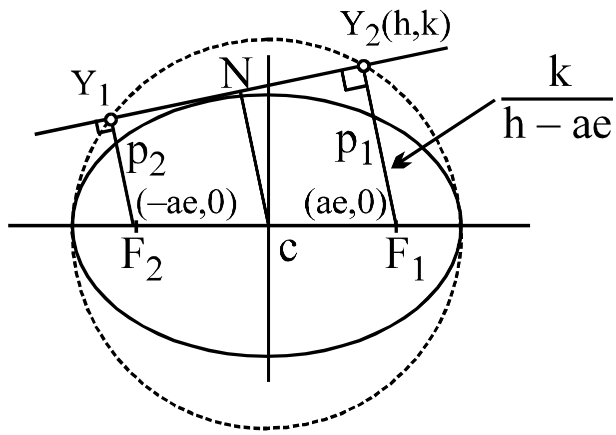
$$= \left| \frac{(a^2m^2+b^2) - m^2a^2e^2}{1+m^2} \right|$$



$$= \left| \frac{a^2m^2 + b^2 - m^2(a^2 - b^2)}{1+m^2} \right|$$

$$= \left| \frac{b^2(1+m^2)}{1+m^2} \right| = b^2$$

(b) Feet of the perpendiculars from either foci on a variable tangent to an ellipse / hyperbola lies on its auxiliary circle. Hence deduce that the sum of the squares of the chords which the auxiliary circle intercept on two perpendicular tangents to an ellipse is constant and is equal to the square on the line joining the foci.



$y = mx + \sqrt{a^2m^2 + b^2}$

$(k - mh)^2 = a^2m^2 + b^2$ **....(1)**

equation of line through F_1 &

slope $= -\frac{1}{m}$

$y - 0 = -\frac{1}{m}(x - ae)$

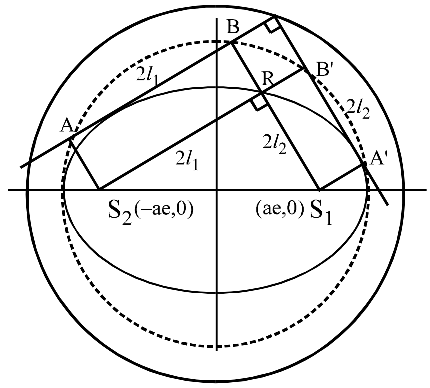
$k = -\frac{1}{m}(h - ae)$

$(km + h)^2 = a^2e^2 = a^2 - b^2$ **....(2)**

adding (1) and (2), we get

$h^2 + k^2 + m^2(h^2 + k^2) = a^2m^2 + b^2 + a^2 - b^2$

$k^2(1 + m^2) + h^2(1 + m^2) = a^2(1 + m^2)$; $x^2 + y^2 = a^2$



now $A'B' = 2l_2$; $AB = 2l_1$

$(S_1R)^2 + (S_2R)^2 = (S_1S_2)^2$.]

(c) If Y_1 and Y_2 be the feet of the perpendiculars on the auxiliay circle from the foci upon any tangent, at P on the ellipse, then the point of intersection 'Q' of the tangents at Y_1 and Y_2 lies on the ordinate through P. If P varies i.e. θ varies then the locus of Q is an ellipse having the same eccentricity as the original ellipse.

Chord of Contact (C.O.C) w.r.t the circle $x^2 + y^2 = a^2$ is

$$hx + ky = a^2 \qquad ....(1)$$

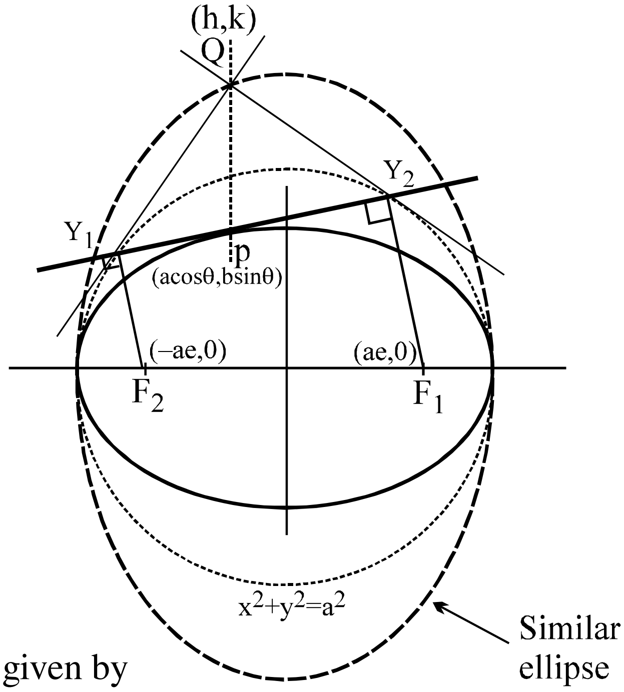
This must be the same tangent at $P(\theta)$

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1 \qquad ....(2)$$

Comparing (1) and (2) $\dfrac{ha}{\cos\theta} + \dfrac{kb}{\sin\theta} = a^2$

$$\left.\begin{aligned} h &= a\cos\theta \\ k &= \frac{a^2\sin\theta}{b} \end{aligned}\right\}$$

$$\frac{x^2}{a^2} + \frac{y^2}{\left(\dfrac{a^2}{b}\right)^2} = 1$$



Which is an ellipse whose eccentricity e' is given by

$$e'^2 = 1 - \frac{a^2b^2}{a^4} \quad \Rightarrow \quad e'^2 = 1 - \frac{b^2}{a^2} = e^2 \ ]$$

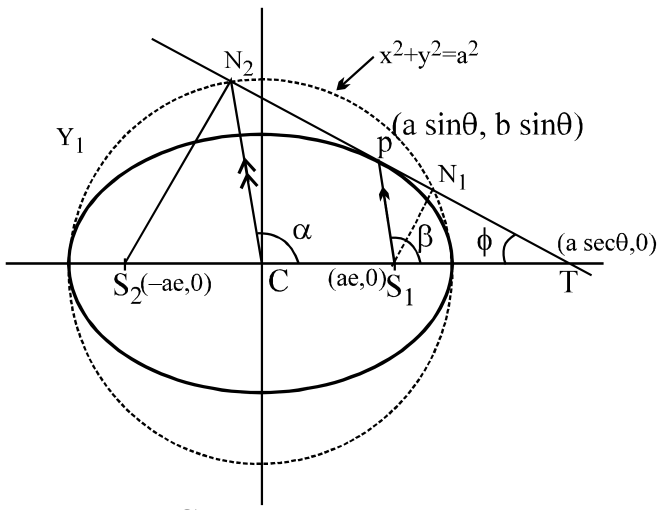
(d) Lines joining centre to the feet of perpendicular from a focus on any tangent at P and the line joining other focus to the point of contant 'P' are parallel.

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

$$\frac{CT}{S_1T} = \frac{a\sec\theta}{a\sec\theta - ae} = \frac{a\sec\theta}{a(\sec\theta - e)}$$

$$\frac{CT}{S_1T} = \frac{1}{1 - e\cos\theta}$$

Again $\dfrac{CN_2}{S_1P} = \dfrac{a}{e\left(\dfrac{a}{e} - a\cos\theta\right)}$



$$= \frac{a}{a - ae\cos\theta} = \frac{1}{1 - e\cos\theta} \Rightarrow \frac{CT}{ST} = \frac{CN_2}{S_1P}$$

Now use sine law in triangles CN_2T and S_1PT we can prove $\alpha + \phi = \beta + \phi$

i.e. $\dfrac{CT}{\sin(\alpha+\phi)} = \dfrac{a}{\sin\phi}$ and $\dfrac{S_1P}{\sin\phi} = \dfrac{S_1T}{\sin(\beta+\phi)}$

$\therefore$ $\dfrac{CT}{a} = \dfrac{\sin(\alpha+\phi)}{\sin\phi}$ and $\dfrac{S_1T}{S_1P} = \dfrac{\sin(\beta+\phi)}{\sin\phi}$

hence $\sin(\alpha + \phi) = \sin(\beta + \phi)$]

H-3 If the normal at any point P on the ellipse with centre C meet the major & minor axes in G & g respectively, & if CF be perpendicular upon this normal, then

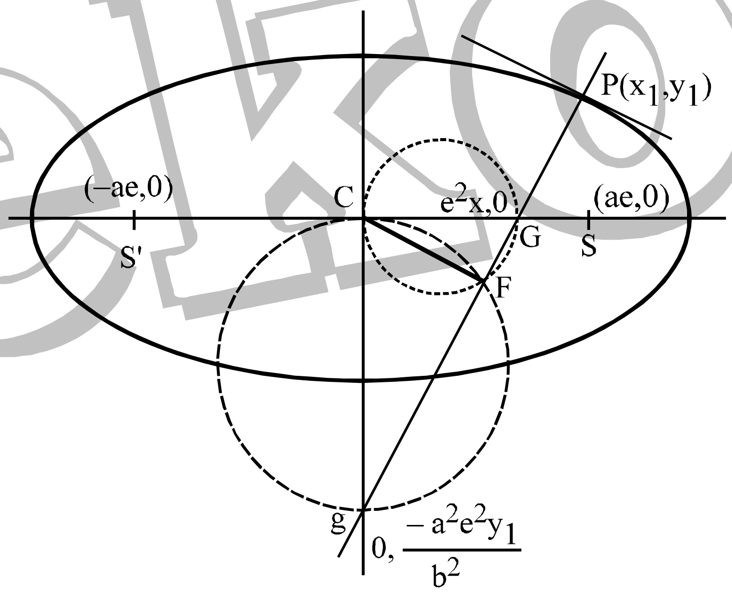
(i) $PF \,.\, PG = b^2$ **(ii)** $PF \,.\, Pg = a^2$

(iii) $PG \,.\, Pg = SP \,.\, S'P$ **(iv)** $CG \,.\, CT = (CS)^2$

(v) locus of the mid point of Gg is another ellipse having the same eccentricity as that of the original ellipse.

[where S and S′ are the focii of the ellipse and T is the point where tangent at P meet the major axis]

(i) $PF \cdot PG = b^2 \; \dfrac{a^2x}{x_1} - \dfrac{b^2y}{y_1} = a^2e^2$



LHS=Power of the point P w.r.t. the circle on CG as diameter

$= x_1(x_1 - e^2x_1) + y_1^2$

$= x_1^2(1 - e^2) + y_1^2$

$= a^2\cos^2\theta \left(1 - 1 + \dfrac{b^2}{a^2}\right) + b^2\sin^2\theta$

$= b^2\cos^2\theta + b^2\sin^2\theta$

$= b^2$]

(ii) $PF \cdot Pg = a^2$

LHS = Power of the point P w.r.t. the circle on Cg as diameter

$$= x_1^2 + y_1\left(y_1 + \frac{a^2e^2y_1}{b^2}\right) = a^2\cos^2\theta + b^2\sin^2\theta\left(1 + \frac{(a^2 - b^2)}{b^2}\right) + b^2\sin^2\theta$$

$$= a^2\cos^2\theta + b^2\sin^2\theta \cdot \frac{a^2}{h^2} = a^2 \;]$$

(iii) PG . Pg = SP . S′P

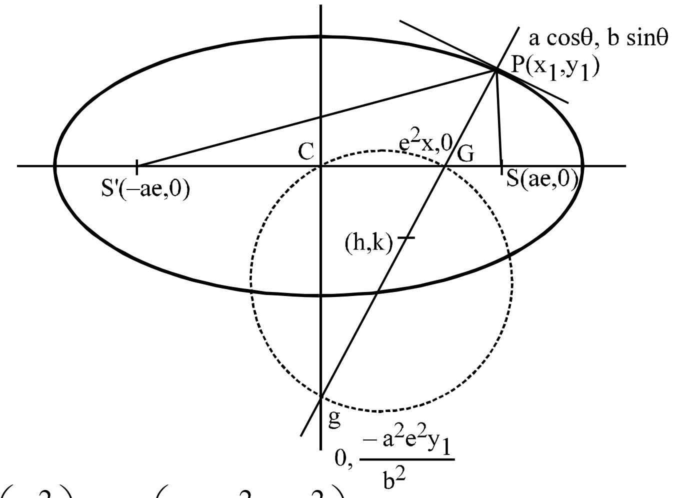
RHS = (a – ae cosθ)(a + ae cosθ)

$a^2 - a^2e^2\cos^2\theta$

$a^2 - (a^2 - b^2)\cos^2\theta$

$a^2\sin^2\theta + b^2\cos^2\theta$

LHS = Power of P w.r.t the circle on Gg as diameter



$$= x_1(x_1 - e^2x_1) + y_1\left(y_1 + \frac{a^2e^2y_1}{b^2}\right)$$

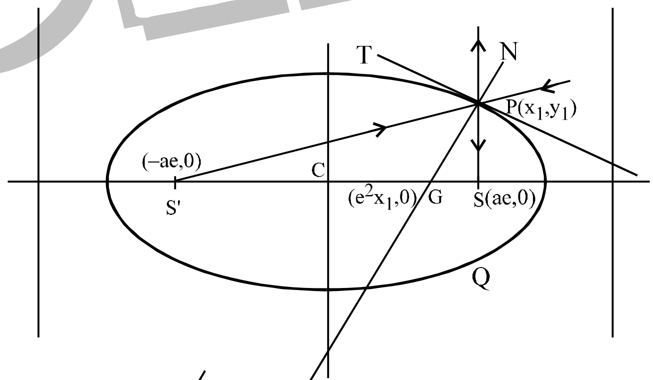
$$= x_1^2(1-e^2) + y_1^2\left(1+\frac{a^2e^2}{b^2}\right) = x_1^2\left(\frac{b^2}{a^2}\right) + y_1^2\left(1+\frac{a^2-b^2}{b^2}\right)$$

$$= b^2\cos^2\theta + a^2\sin^2\theta = \text{RHS} \qquad]$$

(iv) CG . CT = $(CS)^2$

H-4 The tangent & normal at a point P on the ellipse bisect the external & internal angles between the focal distances of P. This refers to the well known reflection property of the ellipse which states that rays from one focus are reflected through other focus & vice−versa. Hence we can deduce that the straight lines joining each focus to the foot of the perpendicular from the other focus upon the tangent at any point P meet on the normal PG and bisects it where G is the point where normal at P meets the major axis.

Reflection property :



$$\frac{S_2G}{GS_1} = \frac{e^2x_1 + ae}{ae - e^2x_1} = \frac{a+ex_1}{a-ex_1}$$

or $$\frac{PS_2}{PS_1} = \frac{e\left(\frac{a}{e}+x_1\right)}{e\left(\frac{a}{e}-x_1\right)} = \frac{a+ex_1}{a-ex_1}$$

Also In ΔS_1PN_1 and ΔN_1PQ

$\angle N_1PQ = \angle N_2PS_1 = \theta$

(neglect property)

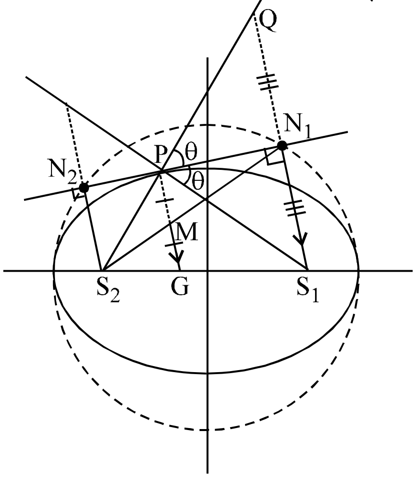
PN_1 is common

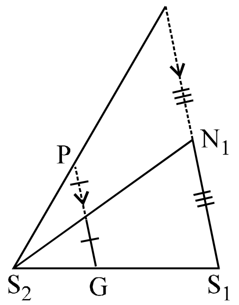
$\angle QN_1P = PN_1S_1 = 90°$

$\therefore \quad S_1PN_1 = N_1PQ_1$

$\therefore \quad N_1$ is the mid point of S_1Q

Now proceed





H-5 The portion of the tangent to an ellipse between the point of contact & the directrix subtends a right angle at the corresponding focus.

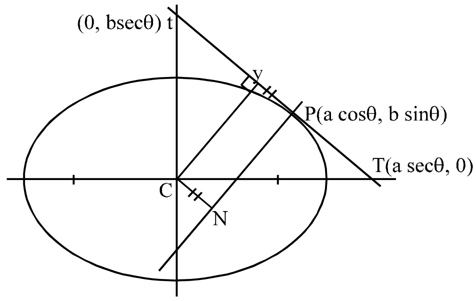
H-6 The circle on any focal distance as diameter touches the auxiliary circle.

H-7 Perpendiculars from the centre upon all chords which join the ends of any perpendicular diameters of the ellipse are of constant length.

H-8 If the tangent at the point P of a standard ellipse meets the axis in T and t and CY is the perpendicular on it from the centre then,

(i) Tt . PY = $a^2 - b^2$ and

(ii) least value of Tt is a + b.



$Tt \cdot py = a^2 - b^2$

$$\sqrt{a^2\sec^2\theta + b^2\operatorname{cosec}^2\theta}$$

$$\sqrt{a^2 + b^2 + (a\sin\theta - b\cos\theta)^2 + 2ab}$$

Home Work : Tutorial Sheet, Ellipse.

TOUGH ELLIPSE

Ex.1 Chords at right angles are drawn through any point P(α) of the ellipse, and the line joining their extremities meets the normal in the point Q. Prove that Q is the same for all such chords, its coordinates being $\dfrac{a^3e^2\cos\alpha}{a^2+b^2}$ and $\dfrac{-a^2be^2\sin\alpha}{a^2+b^2}$.

Prove also that the major axis is the bisector of the angle PCQ, and that the locus of Q for different positions of P is the ellipse :

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{a^2-b^2}{a^2+b^2}\right)^2.$$ [Q.29, Ex-35, Loney]

Ex.2 Prove that the directrices of the two parabolas that can be drawn to have their foci at any given point P of the ellipse and to pass through its foci meet at an angle which is equal to twice the eccentric angle of P. [Q.28, Ex-35, Loney]

Ex.3 An ellipse is rotated through a right angle in its own plane about its centre, which is fixed, prove that the locus of the point of intersection of a tangent to the ellipse in its original position with the tangent at the same point of the curve in its new position is

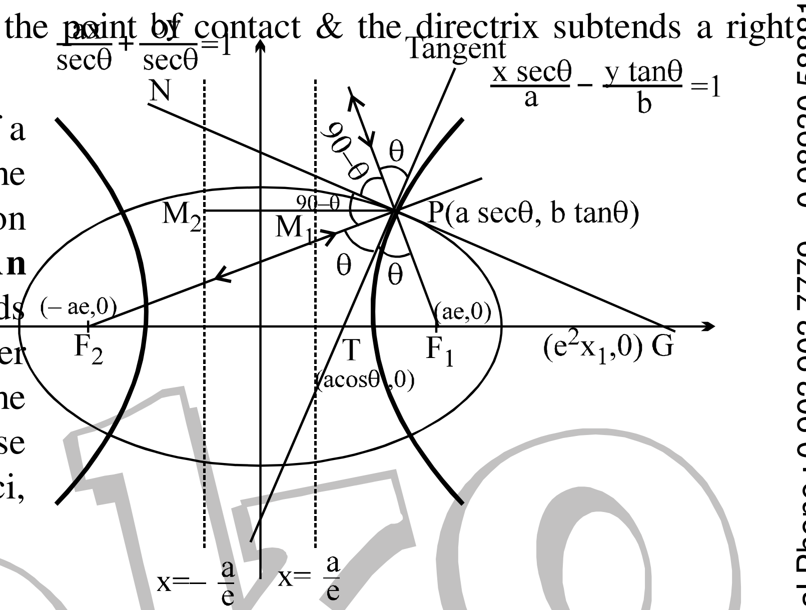
$$(x^2 + y^2)(x^2 + y^2 - a^2 - b^2) = 2(a^2 - b^2)\,xy.$$ [Q.26, Ex-35, Loney

3. HIGHLIGHTS ON HYPERBOLA (TANGENT AND NORMAL) :

H−1 Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ upon any tangent is its auxiliary circle i.e. $x^2 + y^2 = a^2$ & the product of the feet of these perpendiculars is $b^2 \cdot$ (semi C · A)^2

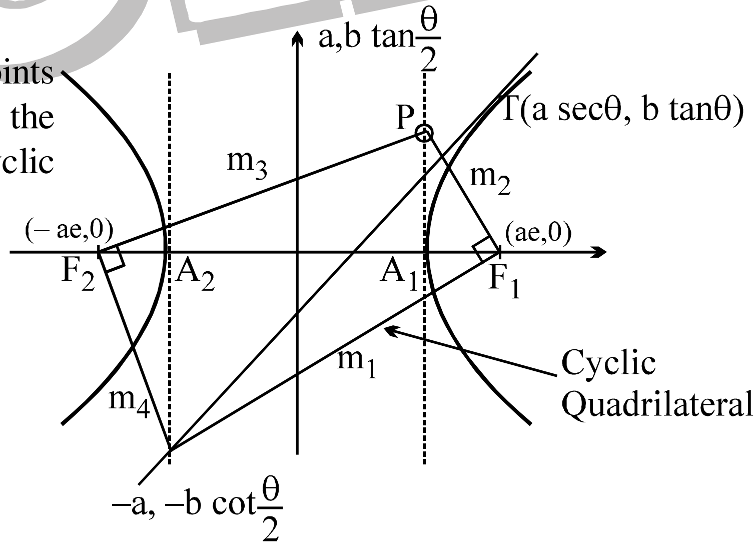
H−2 The portion of the tangent between the point of contact & the directrix subtends a right angle at the corresponding focus.

H−3 The tangent & normal at any point of a hyperbola bisect the angle between the focal radii. This spells the reflection property of the hyperbola as **"An incoming light ray "** aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. It follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common point.



Note that the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{a^2-k^2}-\frac{y^2}{k^2-b^2}=1$ $(a > k > b > 0)$ are confocal and therefore orthogonal.

H−4 The foci of the hyperbola and the points P and Q in which any tangent meets the tangents at the vertices are concyclic with PQ as diameter of the circle.



$$T:\ \frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$$

$$\frac{(\sec\theta-1)b}{\tan\theta}=y$$

$$y = b\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

Family of circles through F_1F_2

$(x - ae)(x + ae) + y^2 + \lambda y = 0 \quad \Rightarrow \quad$ PQ as diameter]

13. ASYMPTOTES :

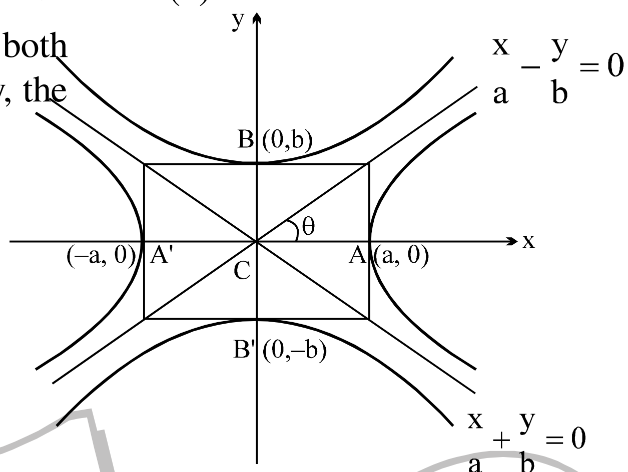
Definition : If the length of the perpendicular let fall from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called the Asymptote of the Hyperbola.

To find the asymptote of the hyperbola :

Let $y = mx + c$ is the asymptote of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Solving these two we get the quadratic as

$(b^2 - a^2m^2)\,x^2 - 2a^2\,mcx - a^2\,(b^2 + c^2) = 0$ ....(1)



In order that $y = mx + c$ be an asymptote, both roots of equation (1) must approach infinity, the conditions for which are :
coeff of $x^2 = 0$ & coeff of $x = 0$.

$\Rightarrow$ $b^2 - a^2m^2 = 0$ or $m = \pm\frac{b}{a}$ &

$a^2\,mc = 0 \Rightarrow c = 0$.

$\therefore$ equations of asymptote are $\frac{x}{a} + \frac{y}{b} = 0$

and $\frac{x}{a} - \frac{y}{b} = 0$.

combined equation to the asymptotes $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

PARTICULAR CASE :

When $b = a$ the asymptotes of the rectangular hyperbola.

$x^2 - y^2 = a^2$ are, $y = \pm x$ which are at right angles.

Note :

(i) Equilateral hyperbola $\Leftrightarrow$ rectangular hyperbola.

(ii) If a hyperbola is equilateral then the conjugate hyperbola is also equilateral.

(iii) A hyperbola and its conjugate have the same asymptote.

(iv) The equation of the pair of asymptotes differ the hyperbola & the conjugate hyperbola by the same constant only.

(v) The asymptotes pass through the centre of the hyperbola & the bisectors of the angles between the asymptotes are the axes of the hyperbola.

(vi) The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis.

(vii) Asymptotes are the tangent to the hyperbola from the centre.

(viii) A simple method to find the coordinates of the centre of the hyperbola expressed as a general equation of degree 2 should be remembered as:

Let $f(x, y) = 0$ represents a hyperbola.

Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$. Then the point of intersection of $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

gives the centre of the hyperbola.

EXAMPLES ON ASYMPTOTES

Ex.1 Find the asymptotes of the hyperbola, $3x^2 - 5xy - 2y^2 - 5x + 11y - 8 = 0$. Also find the equation of the conjugate hyperbola. [Solved Ex. 325 Pg.294]

Ex.2 Find the equation to the hyperbola whose asymptotes are the straight lines $2x + 3y + 3 = 0$ and $3x + 4y + 5 = 0$ and which passes through the point $(1, -1)$. Also write the equation to the conjugate hyperbola and the coordinates of its centre.
[Q.9, Ex-37, Loney]

Ex.3 A normal is drawn to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at P which meets the transverse axis (TA) at G. If perpendicular from G on the asymptote meets it at L, show that LP is parallel to conjugate axis. [Q.7, Ex-37, Loney]

[Sol. equation of GL with slope $-a/b$ and passing through $(e^2x_1, 0)$ is

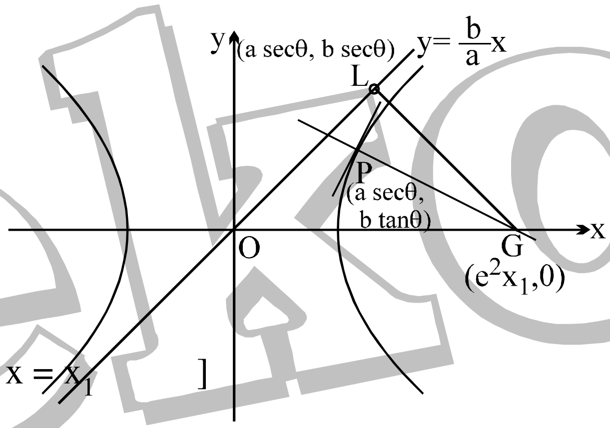
$y - 0 = -a/b(x - e^2x_1)$

$ax + by = ae^2x_1$ ....(1)

put $y = \frac{b}{a}x$

$ax + b\cdot\frac{b}{a}x = ae^2x_1$

$x\left[\frac{a^2+b^2}{a}\right] = a^2e^2x_1$ $\quad x = x_1$]



Ex.4 A transversal cuts the same branch of a hyperbola $x^2/a^2 - y^2/b^2 = 1$ in P, P′ and the asymptotes in Q, Q′. Prove that : (i) PQ = P'Q' & (ii) PQ' = P'Q

[Sol. TPT PQ = P'Q' and PQ' = P'Q

$b^2x^2 - a^2y^2 = a^2b^2$; Let the transversal be $y = mx + c$

$b^2x^2 - a^2(mx + c)^2 = a^2b^2$

$(b^2 - a^2m^2)x^2 - 2a^2mcx - a^2(c^2 + b^2) = 0$

$$\frac{x_1 + x_2}{2} = \frac{2a^2mc}{b^2 - a^2m^2} \qquad ....(1)$$

solving $y = mx + c$ with $b^2x^2 - a^2y^2 = 0$

$(b^2 - a^2m^2)x^2 - 2a^2mcx - a^2c^2 = 0$ ....(2)

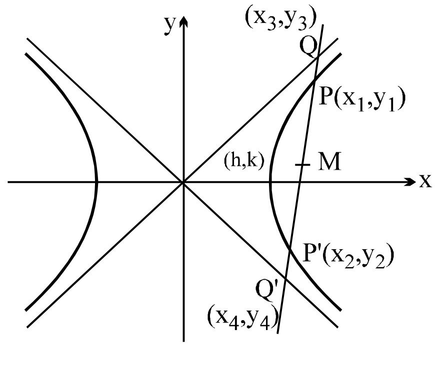
$$\frac{x_3 + x_4}{2} = \frac{2a^2mc}{b^2 - a^2m^2} \qquad ....(3)$$

MQ = MQ'

also MP = MP'

———————

PQ = P'Q']



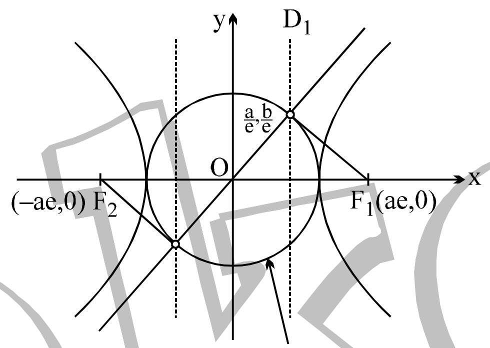
Q.5 The tangent at any point P of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets one of the asymptotes in Q and L, M are the feet of the perpendiculars from Q on the axes. Prove that LM passes through P. [Q.7, Ex.48, (Pillay)]

14. HIGHLIGHTS ON ASYMPTOTES :

H–1 If from any point on the asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point & the curve is always equal to the square of the semi conjugate axis.

H–2 Perpendicular from the foci on either asymptote meet it in the same points as the corresponding directrix & the common points of intersection lie on the auxiliary circle.

$y = \frac{b}{a}x$ ....(1)



$y - 0 = -\frac{a}{b}(x - ae)$

by $+ax = a^2e$ ....(2)

$\left(\frac{b^2}{a} + a\right)x = a^2e$

$\frac{a^2}{e^2} + \frac{b^2}{e^2} = \frac{a^2 + b^2}{a^2} = a^2$

$(b^2 + a^2)x = a \cdot a^2e$

$(a^2e^2)x = a^2ae \quad \Rightarrow \quad x = \frac{a}{e}$; hence $y = \frac{b}{a} \cdot \frac{a}{c} = \frac{b}{c}$]

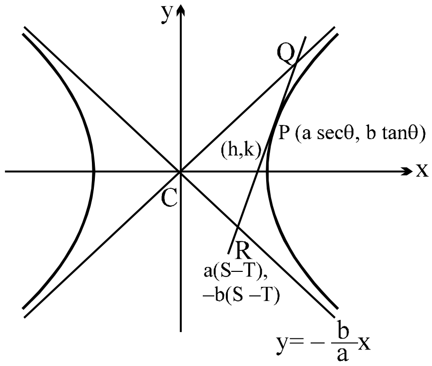
H–3 The tangent at any point P on a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with centre C, meets the asymptotes in Q and R and cuts off a Δ CQR of constant area equal to ab from the asymptotes & the portion of the tangent intercepted between the asymptote is bisected at the point of contact. This implies that locus of the centre of the circle circumscribing the Δ CQR in case of a rectangular hyperbola is the hyperbola itself & for a standard hyperbola the locus would be the curve, $4(a^2x^2 - b^2y^2) = (a^2 + b^2)^2$.

$$\text{Area of } \Delta QCR = \frac{1}{2}\begin{vmatrix} a(S+T) & b(S+T) & 1 \\ 0 & 0 & 1 \\ a(S-T) & -b(S-T) & 1 \end{vmatrix}$$

$$= -\frac{ab}{2}[-(1) - 1] = ab = \text{constant}$$

Area of ΔQCR is constant = ab $\qquad \frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$;

solving it with $y = \frac{b}{a}x$, $\quad \frac{x\sec\theta}{a} - \frac{(\tan\theta)x}{a} = 1$



$x = \frac{a}{S-T} = a(S + T) \quad \Rightarrow \quad x = a(S - T)$

TPT $\;4(a^2x^2 - b^2y^2) = (a^2 + b^2)^2$

$h^2 + k^2 = (h - a(S + T))^2 = 2(S + T)(ah + kb) = 0 \quad ....(1)$

$(S - T)^2(a^2 + b^2) = 2(S - T)(ah - kb) = 0 \quad ....(2)$

$(1) \times (2) \quad \Rightarrow \quad (a^2 + b^2)^2 = 4(a^2x^2 - b^2y^2)$]

H–4 If the angle between the asymptote of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is 2θ then $e = \sec\theta$.

. **RECTANGULAR HYPERBOLA :**

Rectangular hyperbola referred to its asymptotes as axis of coordinates.

(a) Equation is $xy = c^2$ with parametric representation $x = ct$, $y = c/t$, $t \in R - \{0\}$.

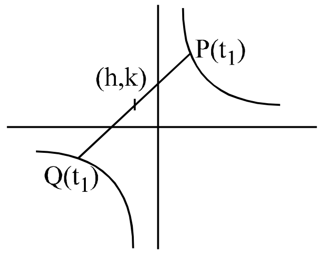
(b) Equation of a chord joining the points $P(t_1)$ & $Q(t_2)$ is $x + t_1 t_2 y = c(t_1 + t_2)$ with slope $m = -\frac{1}{t_1t_2}$

(c) Equation of the tangent at $P(x_1, y_1)$ is $\frac{x}{x_1} + \frac{y}{y_1} = 2$ & at $P(t)$ is $\frac{x}{t} + ty = 2c$.

(d) Equation of normal : $y - \frac{c}{t} = t^2(x - ct)$

(e) Chord with a given middle point as (h, k) is $kx + hy = 2hk$.

[**Explanation** : **(e)** Chord with a given middle point



$2h = c(t_1 + t_2)$

$2k = c\left(\frac{1}{t_1} + \frac{1}{t_2}\right) = c\left(\frac{t_1 + t_2}{t_1t_2}\right) = \frac{2h}{t_1t_2}$; $\qquad \therefore \quad t_1t_2 = \frac{h}{k}$

now equation of PQ is, $\quad y - k = -\frac{1}{t_1t_2}(x - h) = -\frac{k}{h}(x - h)$

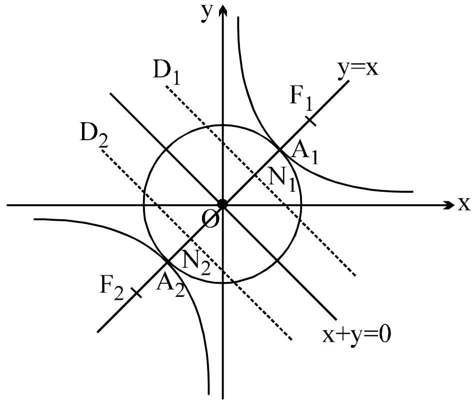
$hy - hk = -kx + hk \quad \Rightarrow \quad kx + hy = 2hk \quad \Rightarrow \quad \frac{x}{h} + \frac{y}{k} = 2$]

(f) Equation of the normal at $P(t)$ is $xt^3 - yt = c(t^4 - 1)$.

Ex.1 Find everything for the rectangular hyperbola $xy = c^2$.

[Sol.1. eccentricity $= \sqrt{2}$ (angle between the two asymptotes = 90°)

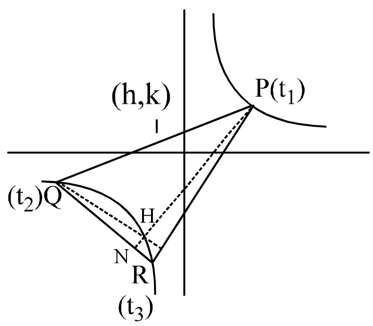
2. asymptotes $x = 0$; $y = 0$



3. T.A. : y = x ; C.A. : y = – x
4. centre (0, 0)
5. vertex (c, c) & (– c, – c)
6. Foci $\left(c\sqrt{2}, c\sqrt{2}\right)$ and $(-c\sqrt{2}, -c\sqrt{2})$
7. Length of Latus rectum = Length of T.A. (in case of rectangular hyperbola) $= 2\sqrt{2}\,c$
8. Equation of auxilliary circle : $x^2 + y^2 = 2c^2$
9. Equation of director circle : $x^2 + y^2 = 0$
10. Equation of the directrices : $x + y = \pm\sqrt{2}\,c$
11. Co-ordinates of the extremities of latus rectum in the 1st quadrant A $\left((\sqrt{2}+1)c, (\sqrt{2}-1)c\right)$

Ex.2 A rectangular hyperbola $xy = c^2$ circumscribing a triangle also passes through the orthocentre of this triangle. If $\left(ct_i, \dfrac{c}{t_i}\right)$ i = 1, 2, 3 be the angular points P, Q, R then orthocentre is $\left(\dfrac{-c}{t_1t_2t_3}, -ct_1t_2t_3\right)$.

[Sol. slope of QR $= -\dfrac{1}{t_2t_3}$



$\therefore$ slope of PN = t_2t_3

$\therefore$ equation of altitude through P

$$y - \frac{c}{t_1} = t_2t_3(x - ct_1)$$

$$y + c\,t_1\,t_2\,t_3 = \frac{c}{t_1} + x\,t_2t_3$$

$$y + c\,t_1\,t_2\,t_3 = t_2t_3\left(x + \frac{c}{t_1t_2t_3}\right) \quad .....(1)$$

(1) is suggestive that orthocentre is $\left(\dfrac{-c}{t_1t_2t_3}, -ct_1t_2t_3\right)$

Ex.3 If a circle and the rectangular hyperbola $xy = c^2$ meet in the four points t_1, t_2, t_3 & t_4, then

(a) $t_1\,t_2\,t_3\,t_4 = 1$

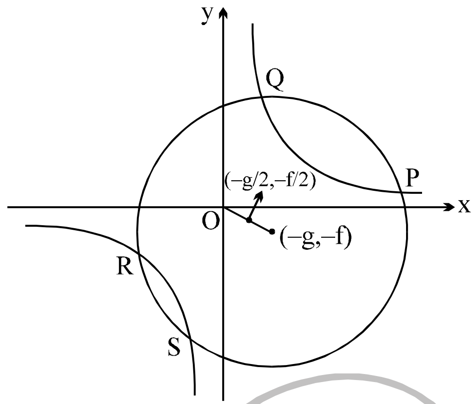
(b) the centre of the mean position of the four points bisects the distance between the centres of the two curves.

(c) the centre of the circle through the points t_1, t_2 & t_3 is :

$$\left\{\frac{c}{2}\left(t_1+t_2+t_3+\frac{1}{t_1t_2t_3}\right), \frac{c}{2}\left(\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+t_1t_2t_3\right)\right\}$$

(d) If PQRS are the four points of intersection of the circle with rectangular hyperbola then $(OP)^2 + (OQ)^2 + (OR)^2 + (OS)^2 = 4r^2$ where r is the radius of circle.

[Sol.



(a) Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + d = 0 \qquad ....(1)$$

solving with $xy = c^2$

$$x^2 + \frac{c^4}{x^2} + 2gx + 2f \cdot \frac{c^2}{x} + d = 0$$

$$x^4 + 2gx^3 + dx^2 + 2fc^2x + c^4 = 0 \qquad ....(1)$$

from (1) $\quad x_1x_2x_3x_4 = c^4$

$c^4\,[t_1t_2t_3t_4] = c^4 \quad \Rightarrow \quad t_1t_2t_3t_4 = 1 \quad \Rightarrow \quad$ (a)

(b) again, centre of the mean position of 4 points of intersection $= \dfrac{\sum x_i}{4}, \dfrac{\sum y_i}{4}$

now from (1)

$$x_1 + x_2 + x_3 + x_4 = -\,2g \qquad .....(2); \qquad \text{hence } \frac{\sum x_i}{4} = -\frac{g}{2}$$

using $xy = c^2$

$$y_1 + y_2 + y_3 + y_4 = c^2\left[\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right] = \frac{c^2}{x_1x_2x_3x_4}\sum x_1x_2x_3 = \frac{c^2}{c^4}(-2f\,c^2) = -2f$$

$$\therefore \quad \frac{\sum y_i}{4} = -\frac{f}{2} \quad ; \quad \text{Hence } \left(\frac{\sum x_i}{4}, \frac{\sum y_i}{4}\right) = \left(-\frac{g}{2}, -\frac{f}{2}\right)$$

(c) centre of the circle through PQR i.e. $(-\,g, -\,f)$ is given by

$$\frac{x_1+x_2+x_3+x_4}{2}, \ \frac{y_1+y_2+y_3+y_4}{2} \quad (\text{using } t_1t_2t_3t_4 = 1)$$

$$\frac{c}{2}\left[(t_1+t_2+t_3)+\frac{1}{t_1t_2t_3}\right], \ \frac{c}{2}\left[\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+\frac{t_1t_2t_3}{1}\right]$$

(d) $(OP)^2 + (OQ)^2 + (OR)^2 + (OS)^2 = 4r^2$ where $r = g^2 + f^2 = d$

LHS $\left(x_1^2+x_2^2+x_3^2+x_4^2\right) + \left(y_1^2+y_2^2+y_3^2+y_4^2\right)$

$$\left[\left(\sum x_1\right)^2 - 4\sum x_1x_2\right] + c^4\left[\frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2} + \frac{1}{x_4^2}\right]$$

$$(4g^2 - 4d) + c^4\left[\left(\sum \frac{1}{x_1}\right)^2 - 2\sum \frac{1}{x_1x_2}\right]$$

$$(4g^2 - 4d) + c^4\left[\left\{\frac{1}{x_1x_2x_3x_4}\sum x_1x_2x_3\right\}^2 - \frac{4}{x_1x_2x_3x_4}\sum x_1x_2\right]$$

$$(4g^2 - 4d) + c^4\left[\left\{\frac{1}{c^4}(-2fc^2)\right\}^2 - \frac{4d}{c^4}\right]$$

$$= (4g^2 - 4d) + (4f^2 - 4d) = 4[g^2 + f^2 - d] = 4r^2 \qquad]$$